ON SOLUTION OF VARIATIONAL PROBLEMS OF ONE-DIMENSIONAL MAGNETOHYDRODYNAMICS

(K RESHENIIU VARIATSIONNYKH ZADACH ODNOMERNOI MAGNITNOI GIDRODINAMIKI

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For calculation of one-dimensional flow of a conducting medium at small magnetic Reynolds numbers it is essential to know the shape of the channel and the distribution of electric and magnetic field intensities. At the present time a large number of references are available which are devoted to the examination of various specific examples. However in the investigation of flow in the channel of a magnetohydrodynamic generator, those problems are of greater interest in which the shape of the channel and the electromagnetic field are selected so as to assure extremes of certain characteristics, for example, maximum available power, minimum losses, etc. The present paper is devoted to solution of these problems with utilization of the method of variational computation. Solutions are illustrated by examples.

Attempts available in the literature to solve the problem under examination either have not led to constructive results [1 and 2] or have a limited value because only some narrow classes of flows, for instance isothermal [3], were examined.

1. Stationary flow of an inviscid and thermally nonconducting medium with electrical conductivity σ° is examined in a flat channel (Fig.1) in the presence of an external magnetic field $\mathbf{B}^\circ = (0, 0, -B^\circ)$. The upper and the lower walls of the channel have at $x^\circ \geqslant 0$ the potentials φ° and $-\varphi^\circ$, respectively. For $x^\circ < 0$ the walls of the channel represent insulators and $B^\circ \equiv 0$. The gas flows from a receiver where it has the density ρ_*°, the enthalpy h_*° and the electrical conductivity σ_*°. It is considered that for $x^\circ < 0$ the flow proceeds without losses.

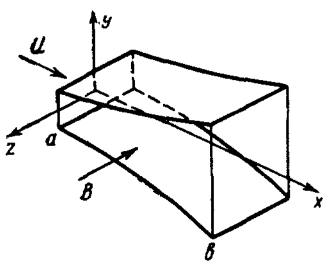

Fig. 1

Assuming that the flow is one-dimensional and that magnetic Reynolds numbers are small, the usual form of Ohm's law is applicable and, neglecting the current which flows parallel to the axis of the channel, we find that the flow is described by equations of motion, energy and continuity [1 and 4]

$$L_1 \equiv \rho u u' + p' + \Delta\sigma B\left(uB - \frac{\varphi}{y}\right) = 0 \qquad \left(\Delta = \frac{B_m^{\circ 2}\sigma_s^\circ l^\circ}{\rho_s^\circ \sqrt{2h_s^\circ}}\right) \tag{1.1}$$

$$L_2 \equiv \left[yu\left(\frac{\varkappa}{\varkappa-1}p + \frac{\rho u^2}{2}\right)\right] + \Delta\sigma\varphi\left(uB - \frac{\varphi}{y}\right) = 0, \qquad L_3 \equiv (y\rho u)' = 0$$

Here $2y$ is the height of the channel, u is the velocity, p is the pressure, Δ is a nondimensional parameter, primes denote derivatives with respect to x, quantities with the superscript $^\circ$ are dimensional, and without this supescript are nondimensional. Dimensional and nondimensional variables are connected through the following relationships:

$$x = \frac{x^\circ}{l^\circ}, \quad y = \frac{y^\circ}{y_a^\circ}, \quad u = \frac{u^\circ}{\sqrt{2h_s^\circ}}, \quad \rho = \frac{\rho^\circ}{\rho_s^\circ}$$

$$p = \frac{p^\circ}{2\rho_s^\circ h_s^\circ}, \quad \sigma = \frac{\sigma^\circ}{\sigma_s^\circ}, \quad B = \frac{B^\circ}{B_m^\circ}, \quad \varphi = \frac{\varphi^\circ}{y_a^\circ B_m^\circ \sqrt{2h_s^\circ}}$$

Here l° and B_s° are constants with dimensions of length and magnetic field intensity, subscripts $a, b, \ldots$ are added to parameters at corresponding points (an exception are subscripts m and s). In writing the energy equation it was assumed that the medium is a perfect gas with an adiabatic index $\varkappa$.

It may be seen from (1.1) that for determination of flow it is necessary to prescribe the controlling parameters: length of channel x_b, its shape $y(x)$, magnetic field $B(x)$, potential $\varphi(x)$ and pressure p_∞ of external medium into which the exhaust takes place. To each set of these quantities corresponds a value of the available power per unit of width of the generator

$$N = \frac{N^\circ}{2\rho_s^\circ (2h_s^\circ)^{3/2} y_a^\circ} = \Delta \int_0^{x_b} \sigma\varphi\left(uB - \frac{\varphi}{y}\right)dx \tag{1.2}$$

Let us examine the problem of determination of $y(x)$, $B(x)$, $\varphi(x)$ x_b and p_∞, which yield a maximum value for functional N.

Variable functions must satisfy conditions connected with formulation of the problem and with limits of applicability of Equations (1.1).

The initial cross section of the channel is fixed as

$$y_a = y(0) = 1, \qquad x_a = 0 \tag{1.3}$$

Maximum allowable dimensions are also given: height $2Yy_a^\circ$ and length l°. Then

$$y(x) \leqslant Y, \qquad 0 \leqslant x \leqslant x_b \leqslant 1 \tag{1.4}$$

Possibilities of arrangements producing the magnetic field, limit the maximum allowable intensity. Taking the modulus of this quantity to be B_s°, we obtain

$$-1 \leqslant B(x) \leqslant 1 \tag{1.5}$$

In an analogous manner

$$-\varphi_m \leqslant \varphi(x) \leqslant \varphi_m \tag{1.6}$$

Finally, by virtue of the assumption of absence of losses, ρ_a and p_a are at $x \leqslant 0$ connected with u_a through the following relationships

$$\rho_a = (1 - u_a{}^2)^{1/(\varkappa-1)}, \qquad p_a = \frac{\varkappa - 1}{2\varkappa}(1 - u_a{}^2)^{\varkappa/(\varkappa-1)} \tag{1.7}$$

Among conditions connected with limits of applicability of Equations (1.1) we will examine only one

$$|y'(x)| \leqslant k < \infty \tag{1.8}$$

where k is a given constant. This condition reflects the circumstance that for one-dimensional equations to be applicable, the angle between the wall and the axis of the channel must not be too great.

Statements made above allow to determine the class of admissible functions. Functions $B(x)$ and $\varphi(x)$ may have discontinuities of the first kind. Function $y(x)$ is continuous in view of (1.8). Assuming the absence of shock waves we obtain from (1.1) that $u(x)$, $\rho(x)$ and $p(x)$ are also continuous although their derivatives are discontinuous at points of discontinuity y', B and φ

The need may arise for additional limiting conditions of the type (1.8). For example, in order to assure continuity of u, ρ and p in supersonic flow it is necessary to require the absence of points of contour discontinuity, i.e. it is necessary to place a restriction on $y''(x)$. Conditions for smallness of magnetic Reynolds number, etc. can be formulated in an analogous manner. Without doing this we note that the presence of regions in the solution which are determined by such inequalities indicates the necessity of application of equations which are valid over a wider range.

Let us formulate the variational problem. It is required to find among permissible functions

$$y = y(x),\ u = u(x),\ \rho = \rho(x),\ p = p(x),\ B = B(x),\ \varphi = \varphi(x)$$

which satisfy conditions (1.3) to (1.8) and differential relationships (1.1). those, which yield a maximum for functional (1.2).

Before proceeding to the solution of the variational problem we note that in calculations it is more convenient to use instead of system (1.1) the equivalent system

$$u' = -\frac{\varkappa p u}{y(\varkappa p - \rho u^2)} y' - \frac{(\varkappa - 1)\varphi - \varkappa y u B}{y(\varkappa p - \rho u^2)} \Delta\sigma \left(uB - \frac{\varphi}{y}\right)$$

$$p' = \frac{\varkappa p \rho u^2}{y(\varkappa p - \rho u^2)} y' + \frac{(\varkappa - 1)\rho u(\varphi - yuB) - \varkappa y p B}{y(\varkappa p - \rho u^2)} \Delta\sigma \left(uB - \frac{\varphi}{y}\right) \tag{1.9}$$

$$\rho = \frac{c}{yu} \qquad (c = \rho_a u_a = y_b \rho_b u_b)$$

2. Let electric conductuvity be constant $(\sigma \equiv 1)$. For solution of the problem we put the auxiliary functional

$$I = \int_0^{x_b} \left[\Delta\varphi\left(uB - \frac{\varphi}{y}\right) + \mu_1(x) L_1 + \mu_2(x) L_2 + \mu_3(x) L_3\right] dx$$

where μ_1, μ_2 and μ_3 are variable Lagrange multipliers. It is apparent that for permissible variation, variations of the functionals I and N coincide by virtue of satisfying Equations (1.1).

Let us find the first variation I .

Since admissible functions or their derivatives can suffer discontinuities, we divide the interval of integration into regions of continuity y', B and φ . To obtain all necessary relationships it is sufficient to examine one point of discontinuity d . Parameters to the left (right) of d we will designate by the index minus (plus). In variation the position of d may change. It may be shown that if δx_d is the change in abscissa of point d, then for any variable z we have $\delta z_{d+} = \delta z_{d-} + (z_- - z_+)_d \delta x_d$. Further, taking advantage of liberty in determination of Lagrange's multipliers, we write

$$\mu_{1d+} = \mu_{1d-}, \quad \mu_{2d+} = \mu_{2d-}, \quad \mu_{3d+} = \mu_{3d-} \tag{2.1}$$

Taking into account the aforementioned and also that $\delta x_a = \delta y_a = 0$ by virtue of (1.3), and that δu_a, $\delta \rho_a$ and δp_a are connected through relationships (1.7), we obtain

$$\delta N = \delta I = \int_0^{x_b} (W_1 \delta y + W_2 \delta B + W_3 \delta\varphi + W_4 \delta u + W_5 \delta\rho + W_6 \delta p)\, dx +$$

$$+ (U_- - U_+)_d\, \delta x_d + V_b \delta x_b - (1 - u_a^2)^{\frac{2-\varkappa}{\varkappa-1}} \left(1 - \frac{\varkappa+1}{\varkappa-1} u_a^2\right) \left(\frac{\mu_2}{2} + \mu_3\right)_a \delta u_a +$$

$$+ \left[\mu_2 u \left(\frac{\varkappa}{\varkappa-1} p + \frac{\rho u^2}{2}\right) + \mu_3 \rho u\right]_b \delta y_b + \left[\mu_1 \rho u + \mu_2 y \left(\frac{\varkappa}{\varkappa-1} p + \frac{3}{2} \rho u^2\right) +\right.$$

$$\left. + \mu_3 \rho y\right]_b \delta u_b + \left(\mu_2 \frac{y u^3}{2} + \mu_3 y u\right)_b \delta\rho_b + \left(\mu_1 + \mu_2 \frac{\varkappa}{\varkappa-1} y u\right)_b \delta p_b \tag{2.2}$$

$$U = \Delta\, (\mu_1 B + \mu_2 \varphi - \varphi) \left(\frac{\varphi}{y} - uB\right), \qquad V = \Delta\varphi \left(uB - \frac{\varphi}{y}\right)$$

Here W_i are known functions of y, u, ρ, p, B and of Lagrange's multipliers. Variations entering into (2.2) are not independent. Lagrange's multipliers are selected such that in the expression for δI only variations of controlling parameters remain, i.e. of y, B, φ, x_d, x_b, y_b and p_∞. We will show that this can be done for any flow. Values of μ_1, μ_2 and μ_3 are determined such that in regions of continuity y', B and φ the following holds:

$$W_4 = W_5 = W_6 = 0$$

From this, taking into account expressions for W_i and simple transformations and with utilization of (1.9) we btain

$$\mu_i' = M_i\, (y, u, \rho, p, B, \varphi, \mu_1, \mu_2, y') \qquad (i = 1, 2, 3) \tag{2.3}$$

$$M_1 = \frac{\varkappa u}{\rho u^2 - \varkappa p} \left[\frac{\mu_1 \rho u}{y} y' + \Delta B\, (\mu_1 B + \mu_2 \varphi + \varphi)\right], \qquad M_2 = \frac{1-\varkappa}{\varkappa y} u M_1$$

$$M_3 = \frac{\varkappa - 1}{2y\, (\rho u^2 - \varkappa p)} \left\{\frac{\mu_1 u}{y} \left(\frac{2\varkappa}{\varkappa-1} p + \rho u^2\right) y' +\right.$$

$$\left. + \Delta \left[(\mu_1 B + \mu_2 \varphi + \varphi)\, u^2 B - 2\mu_1 \left(uB - \frac{\varphi}{y}\right) \left(\frac{\varkappa}{\varkappa-1} uB - \frac{\varphi}{y}\right)\right]\right\}$$

For integration of system (2.3), in addition to conditions (2.1) at points of discontinuity it is indispensable to have three more conditions. Their form depends on the flow behavior.

For subsonic exhaust $p_b = p_\infty$ and $\delta p_b = \delta p_\infty$. If the exit velocity is equal to sonic velocity $(\varkappa p_b = \rho_b u_b^2)$ then

$$\delta p_b = \frac{u_b^2}{\varkappa}\delta\rho_b + \frac{2\rho_b u_b}{\varkappa}\delta u_b \tag{2.4}$$

Finally, for completely supersonic flow u_* is the controlling parameter since it can vary due to change in the shape of the channel at $x < 0$. Small variations in remaining controlling parameters have no influence upon u_* in this case. It is noted that in subsonic flow u_* does not depend on the shape of the channel at $x < 0$, and is completely defined by the flow at $x > 0$.

In accordance with this we obtain in the first two cases by equating to zero the coefficient in front of δu_*

$$2\mu_{3a} = -\mu_{2a} \tag{2.5}$$

In addition to this, in subsonic exhaust, we equate to zero coefficients in front of δu_b and $\delta\rho_b$. As a result we obtain

$$\mu_{1b} = -y_b\left(\frac{\varkappa}{\varkappa-1}\frac{p}{\rho u} + u\right)_b\mu_{2b}, \qquad \mu_{3b} = -\frac{u_b^2}{2}\mu_{2b} \tag{2.6}$$

At sonic conditions by eliminating δp_b by means of (2.4) we find in the same manner the relationships

$$\mu_{1b} = -\frac{\varkappa y_b u_b}{\varkappa-1}\mu_{2b}, \qquad \mu_{3b} = -\frac{u_b^2}{2}\mu_{2b} \tag{2.7}$$

For supersonic flow we obtain the following by equating to zero coefficients in front of δu_b, $\delta\rho_b$ and δp_b :

$$\mu_{1b} = \mu_{2b} = \mu_{3b} = 0 \tag{2.8}$$

Thus, Lagrange's multipliers can always be selected such that in the expression for δI only variations of independently variable quantities remain. For this it is sufficient to fulfill conditions obtained above. For any given x_b, $y(x)$, $B(x)$, $\varphi(x)$ and p_∞ the flow is determined by Equations (1.1) or (1.9) and by the following conditions; by $p_b = p_\infty$ for subsonic exhaust, by

$$\varkappa p_b = \rho_b u_b^2 \tag{2.9}$$

for sonic exhaust (the latter is only possible for $p_\infty \leqslant p_b$), by given u_* in supersonic flow and by (1.7) in all cases. Finally, μ_1, μ_2 and μ_3 are determined from (2.3) for conditions: (2.5) and (2.6), or (2.5) and (2.7), or (2.8) for subsonic, sonic and supersonic conditions, respectively. We note that conditions of continuity (2.1) may remain unused if (2.3) is applied over the entire interval of integration. Here continuity of μ_1 is automatically satisfied.

3. Corresponding to selection of Lagrange's multipliers

$$\delta N = \delta I = \int_0^{x_b} (W_1\delta y + W_2\delta B + W_3\delta\varphi)\,dx + (U_- - U_+)_d\,\delta x_d +$$
$$+ V_b\delta x_b + \left[\mu_2 u\left(\frac{\varkappa}{\varkappa-1}p + \frac{\rho u^2}{2}\right) + \mu_3\rho u\right]_b\delta y_b + \left(\mu_1 + \mu_2\frac{\varkappa}{\varkappa-1}yu\right)_b\delta p_\infty -$$
$$-(1-u_a^2)^{\frac{2-\varkappa}{\varkappa-1}}\left(1 - \frac{\varkappa+1}{\varkappa-1}u_a^2\right)\left(\frac{\mu_2}{2} + \mu_3\right)_a\delta u_a \tag{3.1}$$

The component with δp_∞ exists in (3.1) only in subsonic flow, and the component with δu_a only in supersonic flow.

Variations entering into (3.1) are independent. This permits to obtain conditions of the extremum with respect to all controlling parameters and also with respect to each one of them individually. In addition, examining the variation of some quantity at an arbitrary point (for example y), the remaining variations may be considered to be absent.

First of all we will find the optimum p_∞ for subsonic flow. For this we equate to zero the coefficient in front of δp_∞ . Remembering (2.6) we find that this leads to condition (2.9), i.e. among subsonic conditions the optimum is the behavior of sonic exhaust.

Analogously, in the supersonic case the extremum is realized when one of the following conditions is fulfilled.

$$u_a = \sqrt{\frac{\varkappa - 1}{\varkappa + 1}}, \qquad u_a = 1, \qquad (\mu_2 + 2\mu_3)_a = 0 \tag{3.2}$$

In the first case u_a is equal to sonic velocity, in the second case it is equal to maximum velocity. It is recalled that due to formulation of the problem these conditions correspond to the extremum at fixed y_a. It may be shown that the first value u_a realizes an extremum even when the gas consumption and not y_a is fixed. The character of the extremum is determined by comparison of the quantity N for all roots of (3.2).

In order to find the optimum length of the channel it is necessary to equate to zero the factor in front of δx_b. However, if $x_b = 1$, then the allowable $\delta x_b < 0$ and to insure a maximum of N it is sufficient for this factor not to be negative. Thus,

$$V_b \equiv \Delta\varphi_b \left(uB - \frac{\varphi}{y}\right)_b \geqslant 0 \qquad (x_b \leqslant 1) \tag{3.3}$$

where the sign of inequality can apply only for $x_b = 1$. From this follows the natural conclusion: the length of the channel must be chosen such that in the end section the generator mode is achieved.

In the same manner we find the necessary condition of maximum with respect to y_b

$$\left[\mu_2 u \left(\frac{\varkappa}{\varkappa - 1}\, p + \frac{\rho u^2}{2}\right) + \mu_3 \rho u\right]_b \geqslant 0$$

Here the inequality can apply only for $y_b = Y$. For supersonic flow this condition is always satisfied by virtue of (2.8), for subsonic or sonic conditions it takes the following form because of (2.6) or (2.7)

$$\mu_{2b} \geqslant 0 \tag{3.4}$$

For $y_b < Y$ this condition determines the optimum y_b .

Finally, equating to zero the coefficient in front of δx_d , we obtain the necessary condition for extremum at points of discontinuity

$$(U_- - U_+)_d = 0 \tag{3.5}$$

We emphasize that at points of discontinuity of contour no additional conditions arose.

Examination of terms outside the integral in (3.1) gave the conditions for determination of optimal p_∞, u_a, x_b and x_d at arbitrary $y(x)$, $B(x)$ and $\varphi(x)$. An exception is presented by condition (3.4). In fact, variation y_b is not independent, since by virtue of (1.8) it necessitates a change of y for $x < x_b$. For small changes of y_b the contribution due to this is of a higher degree of smallness since y can be varied only over a section of x of the same order as δy_b. This very circumstance permits to consider δy_b as independent in obtaining (3.4). Therefore the case of arbitrary shape of channel the condition (3.4) serves only as a check and not for finding of optimum y_b

For the construction of experimental $y(x)$, $B(x)$ and $\varphi(x)$, just as in obtaining (3.3) and (3.4), we will remember that the desired curves may consist of regions of two-sided and outer extremums. Since in the first mentioned regions the variations are arbitrary, W_1, W_2 or W_3 must go to zero here.

As a result we obtain Equations

$$W_1 \equiv \frac{\Delta}{y(\rho u^2 - \varkappa p)} \left\{\mu_1 \rho u\, (\varkappa - 1) \left(uB - \frac{\varphi}{y}\right)\left(\frac{\varkappa}{\varkappa - 1} uB - \frac{\varphi}{y}\right) + \right.$$

$$\left. + (\mu_1 B + \mu_2 \varphi + \varphi)\left[\varkappa p \left(uB - \frac{\varphi}{y}\right) + \rho u^2 \frac{\varphi}{y}\right]\right\} = 0 \qquad (3.6)$$

$$W_2 \equiv \Delta \left[u\, (\mu_1 B + \mu_2 \varphi + \varphi) + \mu_1 \left(uB - \frac{\varphi}{y}\right)\right] = 0 \qquad (3.7)$$

$$W_3 \equiv \Delta \left[(1 + \mu_2)\left(uB - \frac{\varphi}{y}\right) - (\mu_1 B + \mu_2 \varphi + \varphi)\, y^{-1}\right] = 0 \qquad (3.8)$$

for determination of the shape of channel, intensities of magnetic field and potential, respectively. In obtaining the expression for W_1 the derivatives u', p', μ_1', μ_2' and μ_3' are eliminated by means of (1.9) and (2.3). Absence of derivative y' in W_1 indicates double degeneracy of the problem. Let us mention that the same circumstance follows from the result of [2]. Each of these equations is applied only where y, B and φ, which are to be determined from these equations, satisfy conditions (1.4) to (1.6) and (1.8). In the opposite case an outer extremum is present. Here the corresonding function is equal to the limiting value resulting from (1.4), (1.5), (1.6) or (1.8). Since in these regions permissible variations do not change sign, necessary conditions of maximum N are formulated here as inequalities

$$W_1 \geqslant 0 \quad \text{for} \quad y = Y \qquad (3.9)$$

$$W_2 \operatorname{sign} B \geqslant 0 \quad \text{for} \quad B = \pm 1 \qquad (3.10)$$

$$W_3 \operatorname{sign} \varphi \geqslant 0 \quad \text{for} \quad \varphi = \pm \varphi_m \qquad (3.11)$$

In order to obtain an analogous condition in the region ef with Equation $y' = k$, we vary y' only for $x_e \leqslant x_l \leqslant x \leqslant x_n \leqslant x_f$, and let max $|\delta y'|$ and $|x_n - x_l|$ be quantities of the same order. With accuracy to terms of higher order

$$\delta N = \delta I = \left(\int_{x_l}^{x_n} \delta y' dx\right) \int_{x_l}^{x_f} W_1 dx$$

For permissible $\delta y'$ we obtain by virtue of (1.8)

$$\int_{x_l}^{x_n} \delta y' dx \leqslant 0 \qquad \text{for } y' = k$$

Therefore the necessary condition of maximum N has the form

$$\int_{x}^{x_f} W_1 dx \geqslant 0 \qquad (y' = k,\ x_e \leqslant x \leqslant x_f)$$

For the same reason

$$\int_{x_e}^{x} W_1 dx \geqslant 0 \qquad (y' = -k,\ x_e \leqslant x \leqslant x_f)$$

To satisfy these inequalities it is sufficient (but not necessary) to satisfy (3.9).

Sometimes the class of permissible functions can be narrowed. So, if the walls are ideal conductors, then $\varphi(x)$ = const . Here $\delta\varphi$ also does not depend on x and the experimental φ satisfies the condition

$$\Big(\int_{0}^{x_b} W_3 dx\Big) \operatorname{sign} \varphi \geqslant 0$$

where the inequality is only applicable at $|\varphi| = \varphi_*$.

An analysis of conditions obtained shows the following. For optimum B and φ the only possible discontinuity is their simultaneous change of signs for unchanged absolute value. This solution, however can be rejected because it yields the same value of N as the continuous solution. If φ is given and continuous then optimum B is also continuous.

In a number of cases φ may be given as discontinuous. Moreover, $|B| = 1$ on both sides of the discontinuity or $B(x)$ is discontinuous because of (3.7). If $\varphi(x)$ is sought in the class of sectionally continous functions with prescribed points of discontinuity (sectional electrodes), then φ is determined in all regions from conditions (3.12) by integration only over regions of constant φ . Optimum dimensions of these regions are found from (3.5). Here at points of discontinuity of φ the optimum $B(x)$ is also either discontinuous or $|B| = 1$ on both sides of the discontinuity. This also applies to the case where B and φ interchange places.

In the general case the extremal contour may consist of regions of four types: $y' = Y, y' = k, y' = -k$ and a region of two-sided extremum (3.6). Extremal magnetic field may contain regions of three types: $B = 1$, $B = -1$ and a region of two-sided extremum (3.7). The same thing can be said about the extremum distribution of the potential.

As follows from (3.7) and (2.8), in supersonic flow the end section of the curve $B(x)$ is always a region of an outer extremum. Functions y, B and φ are continuous at all points of contact.

In sections of the channel which are simultaneously regions of two-sided extremum with respect to y, and with respect to B , according to (3.6) and (3.7) $p = \rho\varkappa^2$, i.e. $M = \varkappa^{-1/2}$. Consequently such a case is impossible for supersonic flow. In addition to this it is not necessary to determine μ_3 in supersonic flow because in this case it has no influence on the solution.

It is clear that the conditions found also give solutions of more specific problems, for example, the problem of determination of external $B(x)$ for given shape of channel and given potential. Here from conditions (3.6) to (3.12) only (3.7) and (3.10) are utilized.

In each actual case all possible flow conditions should be examined (subsonic, sonic, supersonic) and in the presence of several maxima the selection should be made according to quantity N . We note that the case of mixed flow which is not examined in this paper undoubtedly is of interest and requires additional investigation.

4. As the first example let us examine the problem of determining the following optimum values: $B(x)$, φ = const , P_∞ and x_b for various values of parameter Δ in the case of a channel of constant cross section. For $x < 0$ the shape of the channel is such that $M_a \leqslant 1$. There are no limitations on φ .

Thus, it is necessary to solve the boundary value problem for five differential equations (1.9) and (2.3) of the first order where $\sigma \equiv y \equiv 1$ and $y' \equiv 0$, for six boundary conditions: (1.7) and (2.5) for $x = 0$ and (2.7) and (2.9) for $x = x_b$. Additional freedom is given by selection of ρ_b or c in the last equation of system (1.9). The quantity x_b is determined from (3.3), while $B(x)$ in accordance with (3.7), is determined by equation

$$B = \varphi \frac{\mu_1 - (1 + \mu_2) u}{2u\mu_1}$$

if $|\varphi[\mu_1 - (1 + \mu_2) u]| \leqslant |2u\mu_1|$, and it is equal to $+1$ or to -1 in the opposite case. The optimum φ is determined by condition (3.12)

$$\int_0^{x_b} W_3\, dx = 0$$

or by an equivalent differential equation $\chi' = W_3$ for boundary conditions $\chi_* = \chi_b = 0$. One of these conditions is satisfied at the expense of the choice in φ .

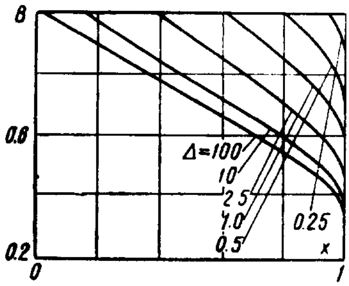

Fig. 2

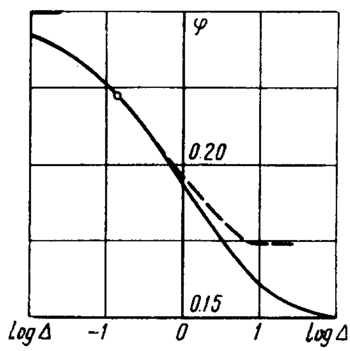

Fig. 3

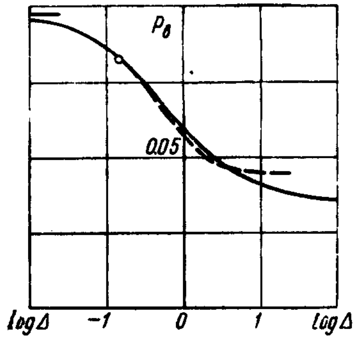

Fig. 4

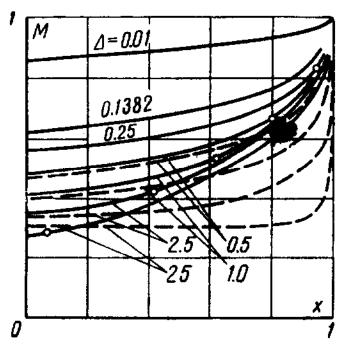

Fig. 5

Equations were integrated by the Kutta-Runge method from $x = x_b$ to $x = 0$. Lacking initial conditions for $x = x_b$ were selected by means of approximations with respect to four parameters using Newton's method. Since on approaching x_b all derivatives tend to infinity, x was taken as independent variable only for $u' < 1$. For $u' \geqslant 1$, u was taken as independent variable.

Calculations were carried out on an electronic computer for $\varkappa = 5/3$ and $0.01 \leqslant \Delta \leqslant 100$. Results are presented in Figs. 2 to 7 by solid lines. In Fig.2 optimum $B(x)$ is shown for a number of values Δ (for all examined Δ the optimum $x_b = 1$). Curves $B(x)$ for $\Delta > 0.1382$ consist of a region with $B \equiv 1$ and a region with two-sided extremum. For smaller Δ the second region is absent. With increasing Δ the extent of the region with two-sided extremum grows, however, for any finite Δ, $B = 1$ near the left end. In Fig.3 the curve for optimum φ is given, and in Fig.4 the curve p_b in its dependence on Δ is given. The optimum (sonic) condition is achieved for $p_\infty \leqslant p_b$. With increase in Δ the actuated pressure drop $\sim p_b^{-1}$ increases and φ decreases, though slower than $\Delta^{-1/2}$. Therefore the dimensional potential increases. In Fig.5 the change in Mach number along the channel is shown for a number of values Δ (circles are points of connection between regions of outer and two-seded extremum).

Fig.6 gives available power as a function of Δ. In Figs.3 to 7 corresponding curves for a generator with $B(x) \equiv 1$ are given by dashed lines, the remaining parameters φ, p_∞ and x_b were optimal. For $\Delta \leqslant 0.1382$ (circles in Figs. 3, 4 and 6) characteristics of both generators coincide. For large Δ optimum profiling of $B(x)$ leads to an increase in available power (by 3.8, 7.1, 22, 31 and 37 per cent for $\Delta = 1.0$, 1.5, 5.0, 10 and 20, respectively) and decrease in φ. In connection with this we note that in the presence of a limitation with

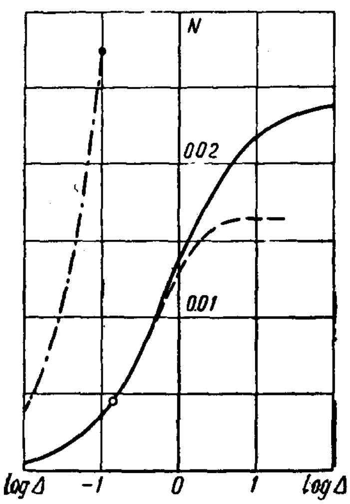

Fig. 6

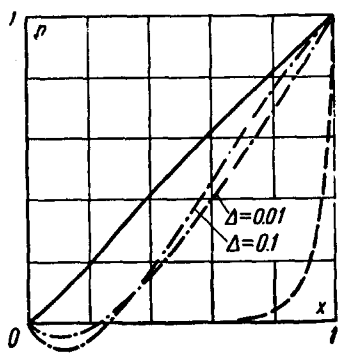

Fig. 7

respect to φ the gain would have been even more significant. In Figs. 3 and 4 optimum φ and p_b for $\Delta = 0$ are shown by horizontal line sections on the left. For determination of $\varphi_{\Delta=0}$ Neuringer's result [1 and 4], $\varphi = u_*/2$ was used, while u_* and p_* were determined from equations of gas dynamics. In accordance with (1.2), $N_{\Delta=0} = 0$. A check of necessary conditions of extremum with respect to y_b and $y(x)$ showed that in cases which were investigated, the channel of the examined formula is not optimal, although $y_b = 1$ is optimal.

It is interesting that for $B \equiv 1$ the region of change of all parameters with increasing Δ becomes constricted towards $x = 1$. This is evident in the Mach number distribution and also in the distribution of available power (in Fig.7, n is the ratio of power which is available in a region of the channel to the left of a given x, to the total power). Such a result is natural because in this case in derivatives in (1.9) a small parameter Δ^{-1} appears. At optimum $B(x)$ the power output is achieved almost uniformly, which confirms qualitative considerations of paper [5].

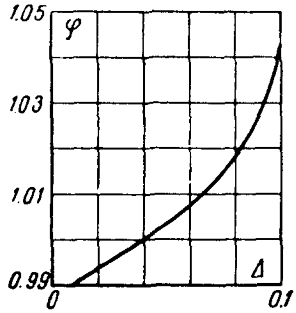

Fig. 8

As a second example the problem of determination of optimum $y(x)$, $B(x)$, φ const and x_b for $l^\circ / y_a^\circ = 10$ and $M_a = 1$ was solved for a number of values Δ and Y in the supersonic flow condition. There is no limitation with respect to φ, and $\varkappa = 5/3$.

In the determination of optimum shape it is necessary to know the constant k or the maximum permissible angle ϑ_m between the wall and the axis of the channel for which one-dimensional theory is still applicable. Since clarification of this peoblem falls beyond the limits of this paper, $\vartheta_m = 20^\circ$, was assumed, this gives

$$k = (l^\circ / y^\circ) \tan \vartheta_m = 3.64$$

and a maximum $Y = 4.64$.

Analysis showed that in the range of Δ and Y under examination, the optimum $x_b = 1$, the optimum magnetic field is uniform: $B(x) = 1$, and the optimum contour of the channel consists of two straight linear sections $y' = k$ and $y = Y$. For $Y = 4.64$ the power of the optimum generator as a function of Δ is given in Fig.6 (dash-dot), and the optimum φ is given in Fig.8. The maximum Δ for which the flow is

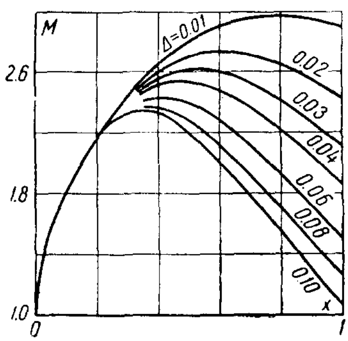

Fig. 9

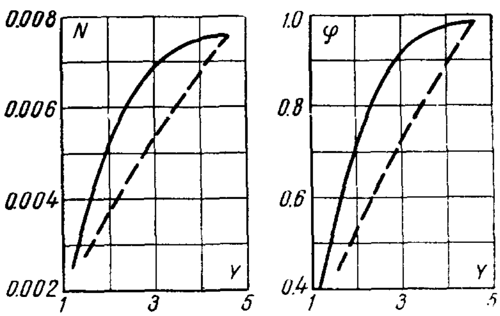

Fig. 10

still supersonic everywhere in this case is equal to 0.103 (black circle in Fig.6). The change in character of flow with increase in Δ is evident in Fig.9. It is interesting that an increase in Δ has almost no influence on the initial region of flow. As follows from Fig.7, where the dash-dot line shows distribution of available power for $\Delta = 0.01$ and 0.1 for $Y = 4.64$, this region of the generator operates as an accelerator. In Fig.10 the dependence of power on Y is given for the optimum generator by a solid curve. The dashed curve is for a generator for which x_b, $B(x)$ and φ are optimum while the walls are formed by straight linear sections which connect the points $x_a = 0$, $y_a = \pm 1$ and $x_b = 1$, $y_b = \pm Y$; both cases $\Delta = 0.02$. In the same figure the dependence $\varphi = \varphi(Y)$ is presented. It is evident that optimum selection of shape leads to a substantial increase in N. Calculation showed that $M_a = 1$ used in this example is not optimum.

5. The analysis made can be applied to a more general case. Let $\sigma = \sigma(\rho, p, \iota)$. In addition it is not always appropriate to carry out optimization according to quantity of available power [6]. In connection with this let us examine the functionals

$$K_j = \int_0^{x_b} \Phi_j(x, y, u, \rho, p, B, \varphi)\, dx \quad (j = 1, \ldots, r-1)$$

$$K_j = \int_0^{x_b} \Phi_j(x, y, u, \rho, p, B, \varphi)\, dx \left[\int_0^{x_b} F_j(x, y, u, \rho, p, B, \varphi)\, dx\right]^{-1} \quad (j = r, \ldots, n)$$

where Φ_j and F_j and also σ are known functions of their arguments.

The variational problem is of interest in which the maximum of the ith functional is sought for isoperimetric conditions which result when the remaining $\sim$ are given. Such is for instance the problem of construction of a magnetohydrodynamic generator of a specified power with minimum Joule dissipation. We construct the function

$$\Phi = \Phi(x, y, u, \rho, p, B, \varphi, \lambda) = \sum_{j=1}^{n} \lambda_j \Phi_j(x, y, u, \rho, p, B, \varphi) - \\ - \sum_{j=r}^{n} \lambda_j K_j F_j(x, y, u, \rho, p, B, \varphi)$$

where λ_j are constant Lagrange's multipliers, here $\lambda_i = 1$, if $i < r$ and

$$\lambda_i = \left(\int_0^{x_b} F_i dx\right)^{-1}, \quad \text{if } i \geqslant r$$

An analysis analogous to the one carried out above again leads to previously obtained relationships if expressions for M_i, W_i, U and V in them are replaced by the following:

$$M_1 = \frac{\varkappa}{\rho u^2 - \varkappa p}\left\{\frac{\mu_1 \rho u^2}{y} y' + \Delta(\mu_1 B + \mu_2 \varphi)\left[u\sigma B - \rho\sigma_\rho\left(uB - \frac{\varphi}{y}\right) - \right.\right. \\ \left.\left. - \sigma_p\left(p + \frac{\varkappa - 1}{\varkappa}\rho u^2\right)\left(uB - \frac{\varphi}{y}\right)\right] + u\Phi_u - \rho\Phi_\rho - \left(p + \frac{\varkappa - 1}{\varkappa}\rho u^2\right)\Phi_p\right\}$$

$$M_2 = \frac{\varkappa - 1}{yu(\varkappa p - \rho u^2)}\left\{\frac{\mu_1 \rho u^2}{y} y' + \Delta(\mu_1 B + \mu_2 \varphi)\left[u\sigma B - (\sigma_p \rho u^2 + \rho\sigma_\rho) \times \right.\right. \\ \left.\left. \times\left(uB - \frac{\varphi}{y}\right)\right] + u\Phi_u - \Phi_p \rho u^2 - \rho\Phi_\rho\right\}$$

$$M_3 = \frac{\varkappa - 1}{2y(\rho u^2 - \varkappa p)}\left\{\frac{\mu_1 u}{y}\left(\frac{2\varkappa}{\varkappa - 1}p + \rho u^2\right)y' - 2\mu_1 \Delta\sigma\left(uB - \frac{\varphi}{y}\right) \times \right. \\ \times\left(\frac{\varkappa}{\varkappa - 1}uB - \frac{\varphi}{y}\right) + \Delta u(\mu_1 B + \mu_2 \varphi)\left[u\sigma B - \rho u^2 \sigma_p\left(uB - \frac{\varphi}{y}\right) - \right. \\ \left.\left. - \frac{\rho\sigma_\rho}{\varkappa - 1}\left(\varkappa - 3 + \frac{2\varkappa p}{\rho u^2}\right)\left(uB - \frac{\varphi}{y}\right)\right] + u\Phi_u - \Phi_p \rho u^2 - \frac{\rho\Phi_\rho}{\varkappa - 1}\left(\varkappa - 3 + \frac{2\varkappa p}{\rho u^2}\right)\right\}$$

$$W_1 = \frac{1}{y(\rho u^2 - \varkappa p)}\left\{\mu_1(\varkappa - 1)\Delta\sigma\rho u\left(uB - \frac{\varphi}{y}\right)\left(\frac{\varkappa}{\varkappa - 1}uB - \frac{\varphi}{y}\right) + \right. \\ + \Delta(\mu_1 B + \mu_2 \varphi)\left[(\sigma\varkappa p - \sigma_p \varkappa p \rho u^2 - \sigma_\rho \rho^2 u^2)\left(uB - \frac{\varphi}{y}\right) + \right. \\ \left.\left. + \rho u^2 \frac{\sigma\varphi}{y}\right] + \varkappa p(\Phi_u u + \Phi_p \rho u^2 - \Phi_y y) - \rho u^2(\rho\Phi_\rho - y\Phi_y)\right\}$$

$$W_2 = \Delta(\mu_1 B + \mu_2\varphi)\left[u\sigma + \sigma_B\left(uB - \frac{\varphi}{y}\right)\right] + \mu_1\Delta\sigma\left(uB - \frac{\varphi}{y}\right) + \Phi_B$$

$$W_3 = \mu_2\Delta\sigma\left(uB - \frac{\varphi}{y}\right) - (\mu_1 B + \mu_2\varphi)\frac{\Delta\sigma}{y} + \Phi_\varphi$$

$$U = \Delta\sigma(\mu_1 B + \mu_2\varphi)\left(\frac{\varphi}{y} - uB\right) + \Phi, \qquad V = \Phi$$

Here Φ_y, Φ_u, Φ_ρ, Φ_p, Φ_B, Φ_φ, σ_ρ, σ_p and σ_B designate partial derivatives. The solution contains as before regions of two-sided and outer extremums. The conclusion about extremal p_∞ and u_* is also retained. Additional freedom in the selection of $(n - 1)$ Lagrange's multipliers serves to satisfy an equal number of isoperimetric conditions.

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